USING DYNAGRAPHS TO INVESTIGATE UNDERSTANDING OF SLOPE

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This paper provides a case study account of a preservice secondary mathematics teacher's thinking while engaging in slope tasks using dynagraphs. The data included audio recordings and screen captures of a small group of preservice teachers engaging with these tasks, with our analysis focusing on the case of Robin. Despite familiarity using slope to measure steepness and determine relationships in a Cartesian plane setting, results indicate Robin struggled to reason about these same uses of slope when presented with a dynagraph. Using the APOS-Slope framework as a lens, the analysis suggests that Robin is limited to thinking of slope as an Action when using slope to measure steepness and determine relationships, relying heavily on shape thinking in the Cartesian plane. Implications for further research are provided.

Keywords: Algebra and Algebraic Thinking; Learning Trajectories and Progressions; Mathematical Representations

Typically introduced in the middle grades (Stanton & Moore-Russo, 2012; Nagle & Moore-Russo, 2014; Nagle et al., 2022), slope serves as a foundational idea for more advanced topics. Slope is used to contrast covariational relationships in linear and nonlinear functions in algebra (Teuscher & Reys, 2010) and serves as a measure of steepness relating to the tangent of an angle in trigonometry (Nagle & Moore-Russo, 2013). Slope is used with regression lines to describe the nature of a data set in statistics (Nagle et al., 2017) and is central to a number of topics in calculus (Asiala et al., 1997; Moore-Russo & Nagle, in press; Zandieh & Knapp, 2006).

One way that slope can be represented is with dynagraphs. Dynagraphs are dynamic representations of functional relationships that allow students to manipulate the value of the input to see the corresponding value of the output, supporting covariational thinking (Lisarelli 2017; Ozen et al. 2021). Dynagraphs use parallel number lines for input and output axes, rather than the traditional perpendicular axes in the Cartesian coordinate plane. Because inputs and outputs are depicted separately, rather than combined in a coordinate pair (Bailey et al. 2020), dynagraphs help direct students' attention to how the two change in relation to each other.

Literature Review

Students need a robust understanding of slope built on covariational reasoning to understand many topics, but even university-level students often have a limited understanding of linear functions and are unable to transition between their different representations (Adu-Gyamfi & Bossé, 2014; Newton, 2018). Some students rely on shape thinking, where a linear graph is treated as a static object (Moore & Thompson, 2015), and slope is perceived to be its visual tilt. These students may incorrectly compute or compare slopes when non-homogenous axes are used (Zaslavsky et al., 2002). Other students reduce slope to mnemonics like "change in *y* over change in *x*" (Walter & Gerson, 2007) and rely on symbolic procedures to calculate a value for slope.

Research provides evidence that students and teachers can fail to make connections between various interpretations of slope (Frank & Thompson, 2021; Hattikudur et al., 2011; Hoban, 2021; Planinic et al., 2012). Students without a robust understanding may think differently about slope depending on the representations (De Bock et al., 2015; Tanışlı & Bike Kalkan, 2018) or the problem contexts (Byerley & Thompson, 2017) within the task, failing to reason conceptually on

application tasks (Lingefjärd & Farahani, 2017) involving slope. Some suggest that this may be attributed to teachers reducing slope instruction to rote procedures (Stump, 1999), and others have reported that pre- and in-service teachers can have limited understandings of slope (Mudaly & Moore-Russo, 2011; Avcu & Türker Biber, 2022; Coe, 2007; Stump, 2001).

Theoretical Framing

APOS theory serves as framing for this paper in terms of a student's development of mathematical knowledge through four stages: Action, Process, Object, and Schema (Dubinsky, 2014). How students engage with mathematical tasks depends on their stage of understanding of the topic involved. We now describe the four APOS stages.

An *Action* is not connected to other mathematical knowledge but instead involves applying a rote procedure or a memorized fact. Actions are typically associated with specific mathematical representations. Once a student repeats and reflects on an Action, it may be internalized so that the student can perform the same transformation as the Action in his or her mind without the need of external stimuli. A *Process* involves meaningful links to other knowledge that allow the student to imagine the transformation, omitting steps that were necessary in the Action stage. These connecting links allow the student to work with different mathematical representations. As the student is able to extend past the Process to deal with new situations, then the Process has been encapsulated into a mental Object. At the *Object* stage, students are able to extend across, and even past, the different representations of a topic to consider how it may apply in novel contexts. Once a mental framework is constructed so that these Actions, Processes, and Objects form an organized, coherent collection, the individual has constructed a *Schema* for the topic.

One may consider the Transition level from the Action to the Process stage (Arnon et al., 2014). On one slope task a student might exhibit behavior in the Process stage but on a slightly different slope task this same student might appear to be at the Action stage. Such is the case at the *Transition* level when students often seem to be moving back and forth between both the Action and Process stages. The APOS-slope framework (Nagle et al., 2016, 2019) is illustrated in Figure 1 below. It focuses on the Action stage, the Transition level, the Process stage, and the Object stage for students who are developing an understanding of slope.

At the Action stage, The APOS-slope framework involves three distinct, isolated notions of slope as a geometric ratio (G), as an algebraic ratio (A), and as a functional property (F). At the Action stage students consider slope to be an intra-representational value that is calculated or identified. The geometric ratio at the Action stage (A_G) involves students who are limited to thinking of slope as a rise-over-run calculation. The algebraic ratio at the Action stage (A_A) represents students who are limited to using the formula $(y_2 - y_1)/(x_2 - x_1)$ as a memorized fact. The functional property at the Action stage (A_F) involves using the phrase "slope is the rate of change of a function" or identifying the coefficient of the x-term in a linear function to be the slope but without any grounding of the underlying covariational relationship between inputs and outputs.

Transitioning from the Action to the Process stage requires moving beyond blindly following a procedure. Students transitioning to the Process stage for geometric ratio (T_G) are realizing that slope is independent of the location or size of the right triangle used to determine the vertical displacement (ΔV) and horizontal displacement (ΔH) of a line. Students transitioning to the Process stage for algebraic ratio (T_A) are realizing that the slope formula is independent of which two points on the line are used, and it may be applied for a general point (x, y) on the line to obtain a calculation involving symbols rather than a number. Students transitioning to the Process stage for functional property (T_F) are realizing that the format of a linear function impacts if the coefficient of the x term can be used to identify the slope. They are starting to

recognize that $y = (-a/b)x + c/b$ and $kax + kby = kc$ (for k, a nonzero constant) have the same slope. In the three transition scenarios, students are moving beyond a single action and starting to repeat the action in order to describe linearity. Students transitioning from the Action to Process stage may also be starting to make connections between A_G, A_A, and A_F as well as T_G, T_A, and T_F.

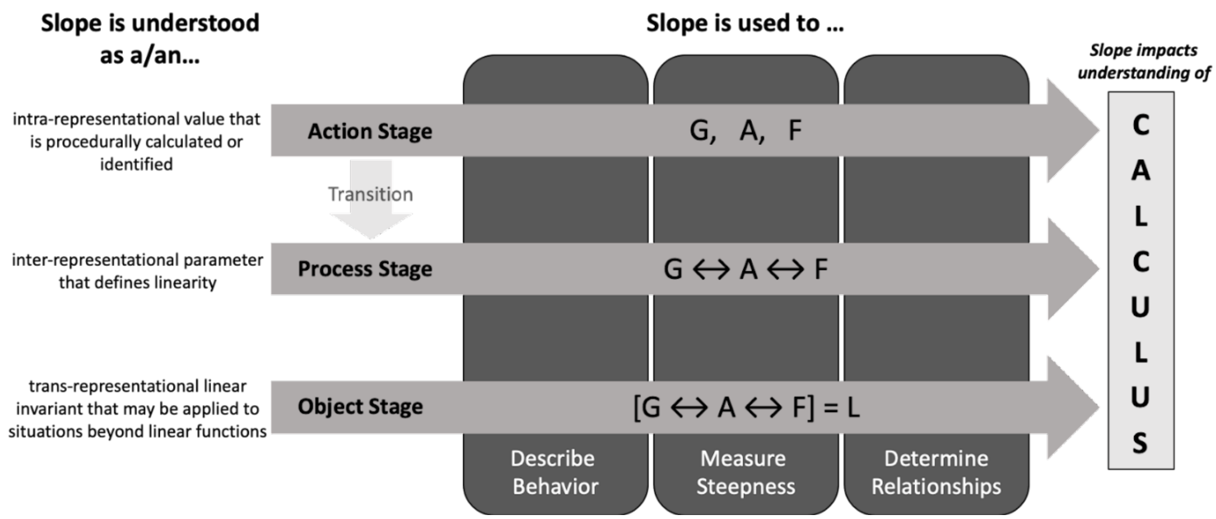


Figure 1: APOS-Slope Framework (Nagle et al., 2019)

At the Process stage, students have an intra-representational understanding of slope that allows them to move easily between visual and analytic representations of slope. They recognize that certain ways of thinking about slope might be more efficient in different contexts, and their understanding of slope is grounded in covariational reasoning (Carlson et al., 2002). Students at the Object stage understand all the Process stage connections. They also are able to view slope as a linear invariant within an equivalence class of ratios that transcends contexts and representations. They can deal with new scenarios that involve unfamiliar units or contexts.

The vertical columns in the APOS-slope framework (see Figure 1) distinguish between the common uses of slope. To *describe behavior* involves using slope to determine if a line's graph or its outputs are increasing, decreasing, or horizontal/constant as corresponding input values increase. To *measure steepness* involves using slope to determine a linear graph's angle of inclination that impacts the severity of its tilt or the outputs' rate of increase when given equal input increments. To *determine relationships* involves using slope to determine if graphs of lines intersect or if systems of linear functions have one, none, or infinitely many solutions. It can also involve using negative reciprocal slopes to determine if a pair of lines intersect in a right angle.

This study leverages dynagraph tasks to evaluate understanding of slope with a preservice teacher. In particular, we consider the following **research question**: *What understanding of slope as a functional property (F) does a preservice teacher evidence while using slope to describe behavior, measure steepness, and determine relationships?*

Methodology

The participants were a convenience sample of a methods class of three preservice teachers at a regional university in the Northeastern US (pseudonyms: Erin, Jessie, Robin). All three were upper-level undergraduate, preservice teachers preparing for student teaching. Our study focuses only on Robin due to space limitations. The researchers modified a traditional slope problem to

develop a series of tasks to capture the different uses of slope (see Figure 2). Each task was accompanied by an interactive dynagraph. One researcher administered the three slope tasks to the group, who worked collaboratively, with accompanying dynagraphs students could manipulate while solving the tasks. The researchers watched interactions independently and completed a line-by-line analysis of each preservice teacher's transcript to determine which slope reasoning in the APOS-Slope Framework was implied and how the dynagraph was leveraged. The two researchers jointly discussed Robin's reasoning, focusing on key shifts in reasoning.

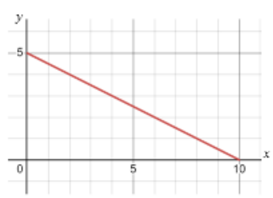
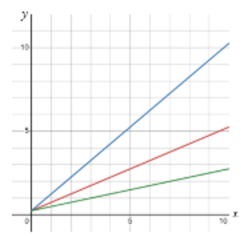
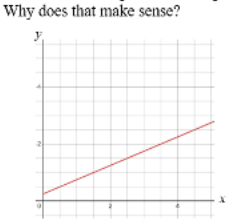
Original Task	*Describe Behavior* Task	*Measure Steepness* Task	*Determine Relationships* Task
It takes Maddy 2.25 hours to mow four lawns and 3.75 hours to mow seven lawns. Find a linear equation that will tell how you how long (y) it takes Maddy to mow x lawns.	1. Explore the relationship between number of lawns and total time spent using the dynagraph at: https://www.geogebra.org/m/yvhevcmg As x increases, what happens to y? Explain why this makes sense given that x is the number of lawns and y is the number of hours it takes Maddy to mow them. As x decreases, what happens to y? Why does this make sense? 2. Now, let's look at a different situation. Suppose that x still represents the number of lawns Maddy mows. Think about a possible scenario where y could represent something other than the number of hours so the relationship between x and y could be represented by the graph below. 	Pokey Paul and Speedy Stef also mow lawns. It takes them the same amount of time as Maddy to get set up. However, Paul takes twice as long as Maddy to mow one lawn while Stef can mow a long in half the time it takes Maddy. Explore Maddy, Paul, and Stef's lawn versus time relationships at https://www.geogebra.org/calculator/dkzrsht6 As x increases, how does y change for each person? For whom does y change *fastest*? For whom does y change *slowest*? Explain why. Three graphs are provided on the same coordinate plane, where x represents number of lawns and y represents time in hours. Please label each one according to whether it represents Maddy, Paul, or Stef. Explain how you know. 	It usually takes Maddy 15 minutes to set-up. Occasionally, small repairs are needed, or gas and oil must be added to the lawn mowers before she starts mowing. This can increase her set-up time. Explore the relationship between the number of lawns mowed and total time spent mowing for Week 1 and Week 2, where you can alter the set-up time for Week 2. https://www.geogebra.org/calculator/pbchf6xe Once Maddy is set-up, does the set-up time impact how quickly she is able to mow lawns? Justify your answer using the dynagraph. The graph representing the amount of time (y) that it takes Maddy to mow x lawns in Week 1 is provided. Sketch two possible graphs for Week 2, one where set-up takes longer than Week 1 and one where set-up is shorter than Week 1. Explain how the graphs are related. Why does that make sense? 

Figure 2: Series of Slope Tasks Given to Participants on Three Uses of Slope

Robin's Results on Three Slope Tasks

We describe Robin's thinking and use of dynagraphs with a number line for number of lawns mowed (input) and parallel number lines for time (outputs) for each of the three tasks. Key elements, including transitions in her thinking, are summarized in Table 1.

***Describe Behavior* Task**

When presented with this task, Robin and her peers began engaging with the dynagraph by dragging the number of lawns mowed to see corresponding changes in the amount of time it takes Maddy to mow. Robin was the first within her group to note that "As x increases, y also increases." After attending to this directional change relationship, the group's discussion turned to the rate at which the time increases. Initially, all three students believed the dynagraph showed that Maddy's time to mow a lawn decreases as the number of lawns increases. Robin used personal experiences, explaining that she "mowed lawns for a summer. It takes practice." She added, "It's taking less time. She's working out how to do it faster or something."

The students manipulated the dynagraph in silence for six seconds, adjusting the number of lawns mowed from smaller to larger values, and back again. Robin suddenly asked, "Oh wait, is it actually taking less time?" to which a peer responded, "Or is it taking the same time?" While it appears, an important cognitive shift may have occurred, Robin then elaborated that she did not "think the line is changing too drastically." She added, "I don't know where it changes" and after

a few seconds of dragging the dynagraph, "It isn't really getting straight." These explanations suggest Robin was looking for cues in the visual representation of the connecting segment linking the input and output in the dynagraph, which might also explain why initially she and her peers thought the rate of change was decreasing. When later prompted by the researcher to support the conclusion about the same-directional relationship between number of lawns and elapsed time, Robin returned to the context of the problem by stating, "Because if you're doing more of something, it can't take you less time. If you mow more lawns, it wouldn't take you less time. It would take you more." Asked to summarize the relationship between lawns mowed and time spent mowing, Robin responded, "They're dependent." This response suggests Robin is coordinating change in inputs and outputs (Carlson et. al., 1996), but her previous discussion also suggests she is coordinating direction of change between inputs and outputs.

When presented with the Cartesian plane graph of a decreasing relationship, one of Robin's peers suggested that the graph could represent the amount of gasoline in the lawn mower, and Robin agreed. At the conclusion of this discussion, Robin reflected on what she learned from the activity, stating, "The relationship between the *x* and *y* axis. The labels on the axis and what they represent. How positive and negative slope have different meanings."

Table 1: Key Elements of Robin's Slope Thinking by Task

	Key Elements of Reasoning	Supporting Quotes	Uses of Dynagraphs	Shifts in Reasoning
Describe Behavior Task	The more lawns mowed; the more time passes.	*As x increases, y also increases.*	Slides inputs back and forth rapidly.	N/A
	The more lawns mowed, the faster she mows each lawn.	*It's taking less time. She's working out how to do it faster or something.*	Continues to slide inputs back and forth.	The rate at which Maddy mows lawns stays the same
Measure Steepness Task	Paul's speed is constant; Stef and Maddy get faster.	*Paul isn't really that slow at the beginning. He's kind of a constant rate, but they [Maddy and Stef] get faster throughout.*	Moves input to 0 where behavior is "consistent" and then to larger values (see Figure 3)	Agrees with peers after discussing the context that *Paul's [output] changes the most.*
Determine Relationships Task	Week 2 takes more time initially, but eventually the time output will match Week 1.	*...as she does more lawns, the setup time doesn't impact as much.* *It catches up eventually.*	Adjusts the set-up time and moves the input to large values (see Figure 4)	*They're never going to pass each other. So, week two's dot will always be further to the right than week one.*

Measure Steepness Task

When this task was posed, Robin was initially quiet as her peers discussed whether Pokey Paul's time changed faster or slower than Speedy Stef's. While the others engaged in this debate,

Robin appeared to be thinking about the relationship she saw, announcing at the first break in conversation that “It’s gonna change somewhere.” She added, “Paul isn’t really that slow at the beginning. He’s kind of at a constant rate. But they [Maddy and Stef] get faster throughout.” Robin’s classmates continued to discuss the relationship between Pokey Paul and Maddy, but Robin appears stuck in her own reasoning. She reiterated, “Paul’s isn’t changing” adding “his slope is like, look at his line throughout the whole thing. It doesn’t really change.” Robin was attending to the slope of the connecting segment between the lawns mowed (input) and Paul’s time spent mowing (output) as shown in Figure 3. Because the input-output connecting segment does not appear to be changing steepness, Robin believed that Paul’s rate was staying the same. On the other hand, Stef’s and Maddy’s connecting segments changed more drastically as the number of lawns mowed increases, leading Robin to explain, “…Paul’s doesn’t change at all, and Maddy’s is changing the slowest, and Stef’s is changing the fastest.”

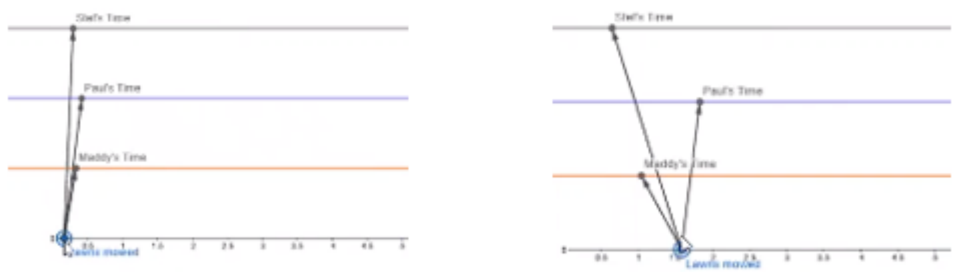


Figure 3: Dynagraph Screenshots that Supported Robin’s Reasoning about Rates

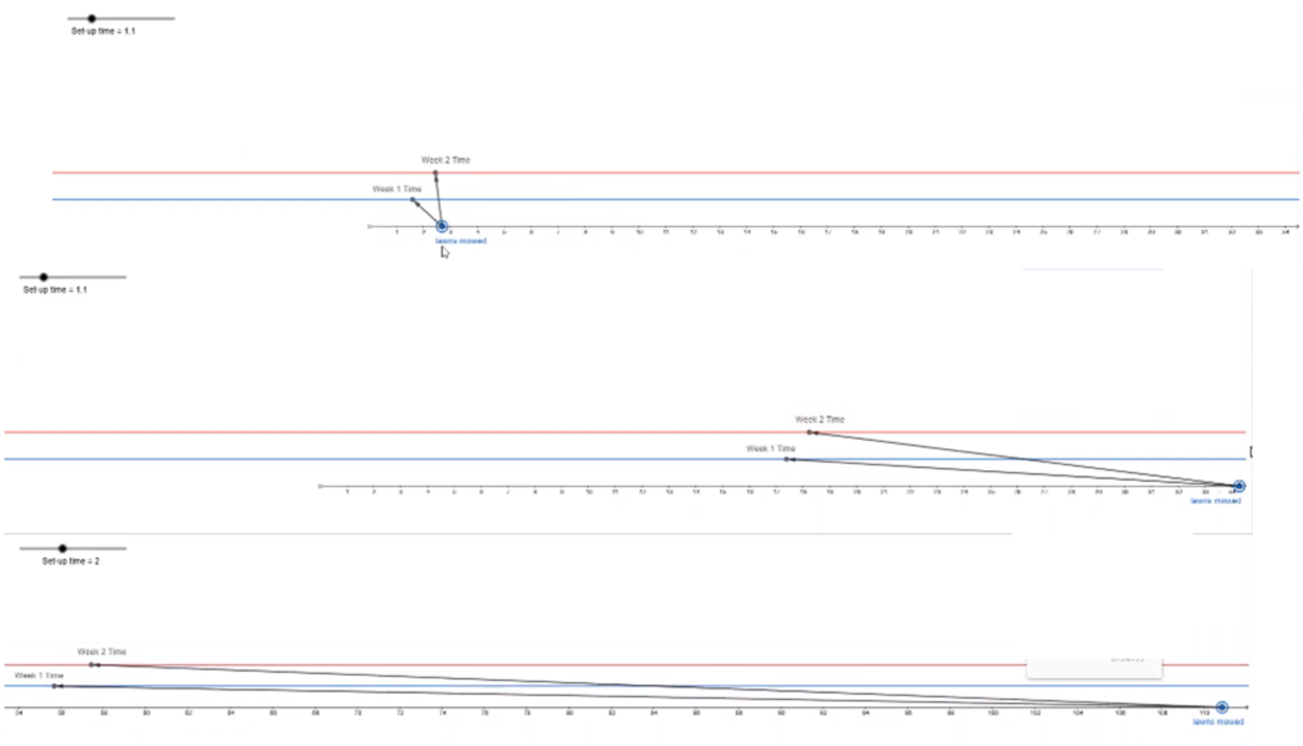
Eventually, one of Robin’s peers applied covariational reasoning about each mowers’ speed and related it to the dynagraph to suggest that Paul’s elapsed time would change the most since it takes him more time to mow the same number of lawns. Although Robin did not acknowledge the dynagraph, she did relate to this contextual explanation, replying, “Oh, I had it backwards. So, Paul’s changes the most.” She and her peers labeled the three Cartesian plane graphs correctly based on steepness and the rate at which each person mows lawns. As the task wrapped up, Robin voiced her frustration with the dynagraph, “I understand the input-output, but it would be easier to look at it, like, as an equation.” In the task reflection she stated, “The relationships matter. For example, just because Paul is slow does not mean his slope takes longer. It is actually a smaller slope.” Her response raises questions about what Robin actually attended to on the dynagraph and how she related it to slope. She knew Paul is the slowest, and that means “Paul’s changes the most.” However, she did not appear to attend to that change via the dynagraph.

***Determine Relationships* Task**

When the task was posed, Robin immediately recognized that set-up time is important stating, “So, if there’s more setup time, then they’re not going to be the same.” But again, the representation in the dynagraph was at odds with her intuition. Robin moved the set-up time to 1.1 hours (“close to 1”) and then discussed what happens as she changed the number of lawns mowed. For a small number of lawns (see Figure 4a), Robin described that “It’s going to obviously be not the same amount,” but as she dragged the number of lawns to larger values (see Figure 4b) she stated that “They almost catch up.” She continued saying, “At the beginning it makes a difference…but as she does more lawns, the setup time doesn’t impact as much.” This belief persisted. A peer explained that since the rate at which Maddy mows the lawns is the

same, Week 2 time will always stay the same amount ahead of Week 1 time. Robin countered, arguing, “It catches up eventually.” She adjusted the set-up time and moved the dynagraph to much larger values of lawns, seemingly looking for a value at which the “catch up” occurred. These interactions illustrate her focus on the connecting segment’s steepness rather than on the rates at which outputs change relative to inputs. Eventually, the researcher posed the question, “Does the set-up time impact the amount of time it takes Maddy to mow lawns?” This prompted more discussion in the group, and Robin adjusted the set-up time to 2 hours once again moving the number of lawns to large values. She conceded, “I guess she probably always is 2 minutes behind, but it is hard to tell when you do the small numbers.” See Figure 4c.

When the group received the Cartesian plane extension problem, Robin very quickly stated, “Yeah, it's going to be parallel because the time it took to mow didn’t change.” Suddenly, Robin recognized that the familiar Cartesian representation should relate to the dynagraph, saying, “Why the heck are these [dynagraphs] so hard to look at? Because you don’t realize that these are technically parallel.” The researcher asked how the parallel relationship was represented in the dynagraph. Robin stated, “The dots. They’re never going to pass each other. So, Week two’s dot will always be further to the right than Week one’s. And then the distance between the two jobs will always stay the same.” Robin verbalized frustration with the dynagraph again, stating, “It would be easier to see on a graph. The lines kind of distract from what you’re actually trying to work out.” In reflection to the activity, Robin reported that the activity helped her to recognize “The idea of having the same slope. Then lines will be parallel. The overall slope won’t change.”



Figures 4a-4c: Series of Screenshots of Dynagraph Manipulations

Discussion

Despite an extensive mathematical background with both constant and variable rates of change, Robin demonstrated limitations in reasoning about slope as a functional property when

presented with a new representation (i.e., dynagraphs). Robin was successful applying functional property thinking to *describe behavior*, “seeing” the corresponding direction of change in inputs and outputs in the dynagraph almost immediately. It was difficult for her to see that the rate of change was constant, but eventually she led her classmates away from the notion that the rate was decreasing while using the dynagraph to observe the input-output relationship. Although other shifts in Robin’s reasoning did occur, this is the only shift that was prompted by the dynagraph and supported by covariational reasoning. This suggests that Robin’s applied at least the T_F stage of reasoning when using slope to *describe behavior*.

Robin had much less success applying functional property reasoning of slope to *measure steepness* and *determine relationships*. The dynagraph representation challenged her familiar visual cues from the Cartesian plane. Robin looked for visual cues in the steepness of the connecting segment linking the inputs and outputs, suggesting that she may be limited to A_G thinking when using slope to *measure steepness*. Her “shape thinking” led her to conclude that Paul’s rate was constant while Stef’s and Maddy’s were speeding up and that eventually the time spent mowing in Week 1 would catch up to the time in Week 2. Despite these misconceptions when working with dynagraphs, Robin was very comfortable using slope when the linear graph was on a Cartesian plane, and she even voiced her frustration at how hard it was to “see” the steepness and parallel relationships in the dynagraph representation. Once the Cartesian plane representation was revealed, Robin made a connection between parallel lines and the distance between Week 1 and Week 2 time values (outputs) staying the same on the dynagraph. Using the two representations in conjunction helped her reason covariationally, an indicator of moving toward T_F thinking while using slope to *determine relationships*. There is no evidence that she was able to reason about slope as a functional property used to *measure steepness* even after engaging with the Cartesian plane representation of the three graphs. One can deduce that Robin’s prior experiences using slope to *measure steepness* and *determine relationships* have been focused primarily on A_G. She may need additional opportunities to think about slope as an algebraic ratio and functional property while using it to *measure steepness* and *determine relationships* in order to develop an understanding of slope as an inter-representational parameter of linearity.

Conclusions

As an unfamiliar representation to Robin, dynagraphs highlighted that she was able to apply functional property reasoning for some, but not all, uses of slope. Despite demonstrating at least T_F reasoning about slope when *describing behavior*, she did not initially apply even A_F reasoning when using slope to *determine relationships* and never evidenced A_F reasoning to *measure steepness*. These results highlight how students’ slope reasoning might vary based on the purpose for which slope is used. The results in this study align with Styers and colleagues’ (2022) findings that teachers need experiences interacting with tasks to build robust notions of steepness through explicit connections to a variety of physical contexts, so they develop imagery and mathematical terminology in rich and meaningful ways. Future slope research should consider which uses of slope are being studied and consider how using multiple representations (e.g., dynagraphs, Cartesian graphs) might provide additional insight into students’ thinking.

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